Light-Cone Quantization Without Periodic Boundary Conditions

Masahiro Maeno

Department of Physics and Earth Sciences, University of the Ryukyus
Senbaru 1 Nishihara cho, OKINAWA 903-0213, JAPAN

This paper describes a light-cone quantization of a two-dimensional massive scalar field without periodic boundary conditions in order to make the quantization manifestly consistent to causality. For this purpose, the field is decomposed by the Legendre polynomials. Creation-annihilation operators for this field are defined and the Fock space was constructed.

1 Introduction

The subject of this paper is the search for a way to carry out light-cone quantization with neither periodic nor anti-periodic boundary conditions.

Recently, much attention has been paid to light-cone quantization. The most remarkable feature of light-cone quantization is the simplicity of its vacuum. Namely, an interacting vacuum is identical to a perturbative vacuum. In light-cone quantization, a mode operator which has a positive (or negative) light-cone momentum(p^+) is respectively an annihilation (or a creation) operator. Due to p^+-conservation, all terms in an interaction Hamiltonian contain at least one annihilation operator. Therefore, the interaction Hamiltonian annihilates the vacuum. However, this situation becomes vague by the existence of so-called 'zero-mode'[1][2][3] which has zero light-cone momentum.

The zero mode plays an important role in light-cone quantization. For example, it can create vacuum structure (see [4][5][6]) or control spontaneous symmetry breaking (see [7][8]). However, the existence of the zero mode crucially depends on which boundary condition we choose.

Let us consider the subject of boundary condition from the perspective of the domain of dependence. Figure 1 shows domains of dependence in equal-time and light-cone quantization without boundary condition.

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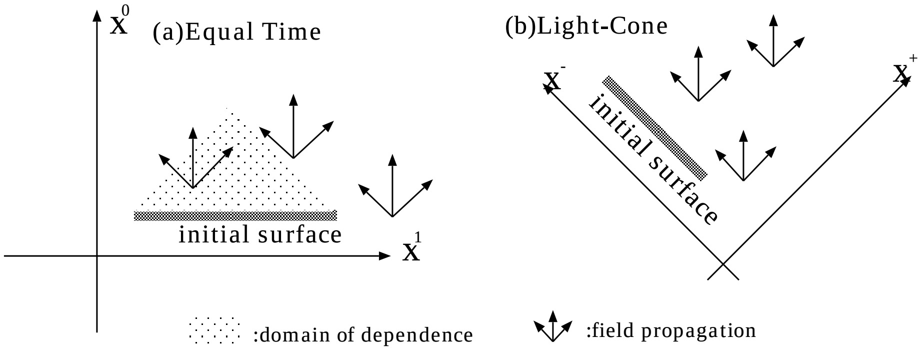
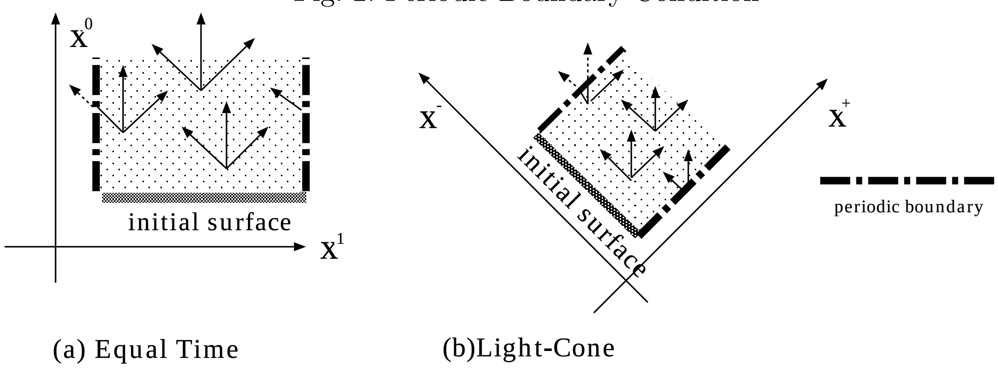
Fig. 1. No Boundary Condition



In light-cone quantization, an initial condition is provided on a constant x^+-plane[1] (in the equal-time case, a constant x^0-plane). Note that we have no domain of dependence for light-cone quantization in this case. This means that we cannot predict the future value of ϕ at any point with initial condition only. Therefore, we set a periodic (or anti-periodic) boundary condition $\phi(x^+, L) = \pm\phi(x^+, -L)$ as Figure 2.

Fig. 2. Periodic Boundary Condition

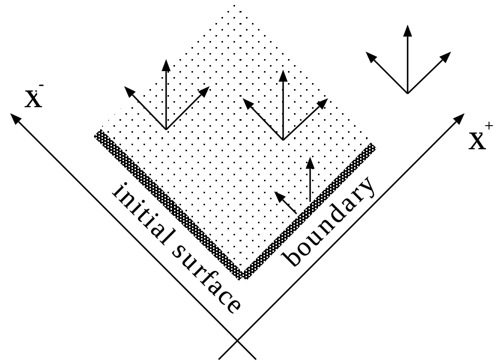


In the equal time quantization case, periodic (or anti-periodic) boundary condition $\phi(x^0, L) = \pm\phi(x^0, -L)$ connects two space-like separated points. This is physically acceptable, but in the light-cone quantization case, the periodic boundary condition is slightly unnatural because it connects two null-like separated points. This prescription of periodic boundary condition is probably acceptable for most of the subjects in light-cone quantization. However, this small problem might be particularly critical for some cases in which boundary information is essential. For this reason, we need to find a method of light-cone quantization with causally natural boundary condition.

For our purpose, the boundary condition for light-cone quantization should be set on the V-shaped boundary as shown in Figure 3. We give an initial condition on a constant x^+-plane($\phi(x^+ = 0, x^-) = \phi_I(x^-)$) and a boundary

[1] Our coordinate convention is $x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right)$. The domain of x^- is $[-L, L]$.

Fig. 3. V-Type Boundary Condition



condition on a constant x^--plane($\partial_+\phi(x^+, x^- = -L) = \partial_+\phi_B(x^+)$). Here, we impose the boundary condition to $\partial_+\phi(x^+, -L)$ in order to prevent a possible inconsistency, as the first candidate of boundary condition $\phi(x^+, x^- = -L) = \phi_B(x^+)$ can conflict with the initial condition $\phi(x^+ = 0, x^-) = \phi_I(x^-)$ on a point $x^+ = 0, x^- = -L$.

2 Legendre Polynomial Expansion

To quantize a field, we need to expand the field operator by some functional basis. Fourier expansion depends strongly on the periodicity; therefore, it is not appropriate to our purpose. Let us expand the field $\phi(x)$ as

$$\phi(x^+, x^-) = \frac{1}{\sqrt{2}} \sum_{m=0}^{2N} a_m(x^+) P_m(x^-) \tag{1}$$

using Legendre polynomials $P_m(x)$. We use a conventional unit:$L = 1$. We can easily recover L by a dimensional analysis whenever desired. To make the following calculation tractable, we limit the mode number m from 0 to $2N$. The number N should be set to infinity after all calculations are complete. At the present stage, no boundary condition is imposed. We will explain below how information of the boundary is introduced to this system.

In the following, we represent a function $f(x)$ defined on $[-1, 1]$ as a ket $|f\rangle$. An integral $\int_{-1}^{1} f^*(x)g(x)dx$ is represented by an inner product $\langle f|g\rangle$. A ket $|x\rangle$ principally denotes an eigenvector of x-representation which satisfies $\langle x|x'\rangle = \delta(x - x')$ and $\int_{-1}^{1} dx|x\rangle\langle x| = 1$.

We define a series of kets $|n\rangle (n = 0, 1, 2, \cdots, 2N)$ by

$$\frac{1}{\sqrt{2}} P_n(x) = \langle x|n\rangle \tag{2}$$

which satisfies

$$\langle k|n\rangle = \int_{-1}^{1} dx \langle k|x\rangle\langle x|n\rangle = \frac{1}{2}\int_{-1}^{1} dx P_k\left(x\right)P_n\left(x\right) = \frac{1}{2n+1}\delta_{kn} \tag{3}$$

$$\sum_n |n\rangle(2n+1)\langle n| = 1. \tag{4}$$

In the following, any Latin index $(k, l, m, n, \cdots)$ repeated in a product is automatically summed from 0 to $2N$. Under this notation, we can write (1) as

$$\phi(x^+, x^-) = a_m(x^+)\langle x^-|m\rangle \tag{5}$$

Defining a derivative operator ∂ as $\frac{\partial}{\partial x}\langle x| = \langle x|\partial$, its derivative can be written as

$$\frac{\partial}{\partial x^-}\phi(x^+, x^-) = a_m(x^+)\langle x^-|\partial|m\rangle. \tag{6}$$

Using well-known formulas for Legendre polynomials, the matrix elements of the operator ∂ are calculated as

$$\langle k|\partial|n\rangle = \begin{cases} 1 & k < n, (k+n) \bmod 2 = 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The Lagrangian is calculated as

$$L_{free} = \int_{-1}^{1} dx^- \left(\partial_+\phi\partial_-\phi - \frac{m^2}{2}\phi^2\right) = \dot{a}_k\langle k|\partial|l\rangle a_l - \frac{m^2}{2}a_k\langle k|l\rangle a_l \tag{8}$$

This expression is non-diagonal with respect to a_n because $\langle k|\partial|l\rangle$ is a non-diagonal matrix. Although it makes the following calculation cumbersome, we have to pay this price for avoiding periodic boundary conditions.

3 Canonical Quantization

In this section, we carry out canonical quantization. As explained in the previous section, we impose the boundary condition $\dot{\phi}(x^+, -L) = \dot{\phi}_B(x^+)$. For this purpose, we add

$$\sqrt{2}B(\dot{\phi}_B(x^+) - \dot{\phi}(x^+, -L)) = \sqrt{2}B\dot{\phi}_B(x^+) - B(-1)^n\dot{a}_n \tag{9}$$

to the Lagrangian density. The new variable B is the Lagrange multiplier (a counterpart of the Nakanishi-Lautrup field) and ϕ_B is a classical boundary value of ϕ.

Conjugate momenta of a_n and B are written as

$$\pi_k = \langle k|\partial|l\rangle a_l - (-1)^k B \tag{10}$$
$$\pi_B = 0. \tag{11}$$

Because all π contain no time(x^+)-derivative, these equations must be treated as constraints.

$$\varphi_k \equiv \pi_k - \langle k|\partial|l\rangle a_l + (-1)^k B = 0 \tag{12}$$
$$\varphi_B \equiv \pi_B = 0 \tag{13}$$

To calculate Dirac brackets, we define new linear combinations of constraints as

$$\tilde{\varphi}_{\tilde{n}} \equiv \varphi_{\tilde{n}} - (-1)^{\tilde{n}}\varphi_0 + \varphi_B\langle 0|\partial|\tilde{k}\rangle. \tag{14}$$

From now on, we use indices with a tilde $(\tilde{k}, \tilde{l}, \tilde{n}, \cdots)$ for numbers running from 1 to $2N$ (exclude 0).

The modified constraints are

$$\tilde{\varphi}_{\tilde{k}} = \pi_{\tilde{k}} - (-1)^{\tilde{k}}\pi_0 - \langle\langle\tilde{k}|\partial|l\rangle a_l + \pi_B\langle 0|\partial|\tilde{k}\rangle \tag{15}$$
$$\varphi_0 = \pi_0 - \langle 0|\partial|k\rangle a_k + B \tag{16}$$
$$\varphi_B = \pi_B \tag{17}$$

where we use a notation $\langle\langle n| \equiv \langle n| - (-1)^n\langle 0|$. The Poison bracket between these constraints is

$$\begin{array}{c c} & \begin{array}{ccc} \tilde{\varphi}_{\tilde{l}} & \varphi_0 & \varphi_B \end{array} \\ \begin{array}{c} \tilde{\varphi}_{\tilde{k}} \\ \varphi_0 \\ \varphi_B \end{array} & \begin{pmatrix} \langle\langle\tilde{k}|\bar{\partial}|\tilde{l}\rangle\rangle & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \tag{18}$$

where $\bar{\partial} = \partial - \partial^\dagger$. The elements of the matrix $\langle\langle\tilde{k}|\bar{\partial}|\tilde{l}\rangle\rangle$ and its inverse are

$$\langle\langle\tilde{k}|\bar{\partial}|\tilde{l}\rangle\rangle = \begin{cases} 2 & \tilde{k} < \tilde{l},\ \tilde{k} \text{ is odd, } \tilde{l} \text{ is even.} \\ -2 & \tilde{k} > \tilde{l},\ \tilde{k} \text{ is even, } \tilde{l} \text{ is odd.} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\langle\!\langle \tilde{j}|\bar{\partial}^{-1}|\tilde{k}\rangle\!\rangle = \frac{1}{2}\left(\delta_{\tilde{j},\tilde{k}+1} - \delta_{\tilde{j}+1,\tilde{k}}\right). \tag{20}$$

All $2N+2$ constraints are second-class. We can calculate the Dirac brackets as

$$\begin{aligned} \{\psi_1, \psi_2\}_{DB} &= \{\psi_1, \psi_2\}_{PB} + \{\psi_1, \tilde{\varphi}_{\tilde{k}}\}_{PB} \langle\!\langle \tilde{k}|\bar{\partial}^{-1}|\tilde{l}\rangle\!\rangle \{\tilde{\varphi}_{\tilde{l}}, \psi_2\}_{PB} \\ &\quad + \{\psi_1, \varphi_0\}_{PB} \{\varphi_B, \psi_2\}_{PB} - \{\psi_1, \varphi_B\}_{PB} \{\varphi_0, \psi_2\}_{PB}. \end{aligned} \tag{21}$$

The results are listed below:

$$[a_k, B]_{DB} = -\frac{1}{2}(\delta_{k,0} + \delta_{k,2N}) \tag{22}$$

$$[a_k, a_l]_{DB} = \frac{1}{2}(\delta_{k+1,l} - \delta_{k,l+1} + \delta_{k,0}\delta_{l,2N} - \delta_{k,2N}\delta_{l,0}) \tag{23}$$

Commutation relations are obtained from the above by multiplying $i\hbar$.

The Hamiltonian has the mass term H_m and the boundary-fixing term H_B as

$$H_m = \frac{m^2}{2} a_k \langle k|l\rangle a_l, \quad H_B = -\sqrt{2} B \partial_+ \phi_B. \tag{24}$$

The time development of the boundary value $\phi(x^+, -L) = \frac{1}{\sqrt{2}} a_n (-1)^n$ is now derived from a commutator with the Hamiltonian as

$$\partial_+ \phi(x^+, -L) = -i \left[\phi(x^+, -L), H_m + H_B\right]_{DB} = \partial_+ \phi_B. \tag{25}$$

The result is consistent to our boundary condition.

4 Creation-Annihilation Operators

In this section, we construct creation-annihilation operators. For simplicity, we consider $\partial_+ \phi_B = 0$ case, which is a light-cone extension of Neumann boundary condition. Let us define a new operator $A(\alpha)$ as a linear combination of a_n:

$$A(\alpha) = \lambda_n(\alpha) a_n \tag{26}$$

which satisfies

$$[A(\alpha), H_m] = \alpha \frac{m^2}{2} A(\alpha) \tag{27}$$

where $\lambda_n(\alpha)$ are c-number coefficients which depend on α and n. By the definition, this operator $A(\alpha)$ lowers the energy eigenvalue by $\alpha\frac{m^2}{2}$. $A(\alpha)$ with positive (or negative) α are annihilation (or creation) operators, respectively.

In the following, we calculate the coefficients $\lambda_n(\alpha)$. From (27), they have to satisfy the following equations:

$$\lambda_1(\alpha)+\lambda_{2N}(\alpha)=i\alpha\lambda_0(\alpha) \tag{28}$$

$$\lambda_{n+1}(\alpha)-\lambda_{n-1}(\alpha)=i\alpha(2n+1)\lambda_n(\alpha) \qquad \text{for } 1\le n\le 2N-1 \tag{29}$$

$$-\lambda_0(\alpha)-\lambda_{2N-1}(\alpha)=i\alpha(4N+1)\lambda_{2N}(\alpha). \tag{30}$$

Solving (28) and (29), all $\lambda_n(\alpha)$s can be written by $\lambda_0(\alpha)$ and $\lambda_{2N}(\alpha)$ as

$$\lambda_m(\alpha)=\sum_{l=0}^{m}\left(\frac{i\alpha}{2}\right)^l\frac{(m+l)!}{l!(m-l)!}\times\left\{\begin{array}{cl}-\lambda_{2N}(\alpha) & \text{for odd } m+l\\ \lambda_0(\alpha) & \text{for even } m+l\end{array}\right\} \tag{31}$$

which can be easily proved by induction.

These equations lead to two expressions for $\lambda_{2N}(\alpha)$.

From (31),

$$\lambda_{2N}(\alpha)=\frac{\sum_{m=0}^{N}\left(\frac{i\alpha}{2}\right)^{2m}\frac{(2N+2m)!}{(2m)!(2N-2m)!}}{1+\sum_{m=0}^{N-1}\left(\frac{i\alpha}{2}\right)^{2m+1}\frac{(2N+2m+1)!}{(2m+1)!(2N-2m-1)!}}\times\lambda_0(\alpha). \tag{32}$$

From the equation (30),

$$\lambda_{2N}(\alpha)=\frac{1+\sum_{m=0}^{N-1}\left(\frac{i\alpha}{2}\right)^{2m+1}\frac{(2N+2m)!}{(2m+1)!(2N-2m-2)!}}{\sum_{m=0}^{N-1}\left(\frac{i\alpha}{2}\right)^{2m}\frac{(2N+2m-1)!}{(2m)!(2N-2m-1)!}-i\alpha(4N+1)}\times\lambda_0(\alpha) \tag{33}$$

Agreement of these two expressions implies a condition for α;

$$\sum_{m=0}^{N}\left(\frac{i\alpha}{2}\right)^{2m+1}\frac{(2N+2m+1)!}{(2m+1)!(2N-2m)!}=0. \tag{34}$$

This equation has $2N+1$ solutions as $\alpha=(0,\pm\alpha_1,\pm\alpha_2,\cdots,\pm\alpha_N)$.[2] At the present stage, all $\lambda_n(\alpha)$ have a factor $\lambda_0(\alpha)$. So we can freely set normalization

[2] Although we do not go into detail, numerical calculation shows that the solutions of (34) are nearly to $\frac{2}{(2n+1)\pi}$. These values are obtained as a result of $N\to\infty$ limit in the next section.

of $\lambda_0(\alpha)$. As a convention, we set $\lambda_0(\alpha)$ to be a real number which satisfies $\lambda_0(\alpha) = \lambda_0(-\alpha)$. In this convention, $\lambda_n(-\alpha) = \lambda_n^*(\alpha)$ and $A(-\alpha) = A^\dagger(\alpha)$.

Now we introduce a set of vectors in the function space as

$$|\alpha\rangle = (2n+1)\lambda_n(\alpha)|n\rangle. \tag{35}$$

$|\alpha\rangle$ is also a set of linear independent $2N+1$ vectors $\{|\alpha\rangle | \alpha = 0, \pm\alpha_1, \pm\alpha_2, \cdots, \pm\alpha_N\}$. As easily confirmed using (28),(29), (30) and $\langle m|n\rangle = \frac{1}{2n+1}\delta_{mn}$, $\langle\alpha|\beta\rangle = 0$ for $\alpha \neq \beta$. We set normalization of $\lambda_0(\alpha)$ to $\langle\alpha|\alpha\rangle = 1$. In this convention, $\lambda_n(\alpha) = \langle\alpha|n\rangle$. We will use this expression from now on. We calculate commutation relation of $A(\alpha)$s as

$$[A(\alpha), A(\beta)] = \alpha\frac{2n+1}{2}\langle\alpha|n\rangle\langle\beta|n\rangle = \frac{\alpha}{2}\delta_{\alpha+\beta,0}. \tag{36}$$

The mass term of the Hamiltonian can be written by $A(\alpha)$ as

$$H_m = \frac{m^2}{2}\sum_\alpha A(-\alpha)A(\alpha) \tag{37}$$

and the field ϕ is expanded as

$$\phi(x^+, x^-) = \sum_\alpha e^{-i\frac{\alpha m^2}{2}x^+}\langle x|\alpha\rangle A(\alpha) \tag{38}$$

Here, the function $\langle\alpha|x\rangle$ plays a role of $e^{ip^+x^-}$ in the periodic boundary condition case.

The Fock space is constructed as

$$[A(-\alpha_1)]^{n_1}[A(-\alpha_2)]^{n_2}\cdots[A(-\alpha_N)]^{n_N}|0\rangle \times (\text{a function of } A(0)) \tag{39}$$

Since the operator $A(0)$ commutes with all other $A(\alpha)$, it does not belong to the creation-annihilation pair. Hence, we use coordinate representation for $A(0)$.

We show that we can define creation-annihilation operators even if we do not use periodic boundary conditions.

5 $N \to \infty$ Limit

Now we take a limit $N \to \infty$. First, we divide the equation (34) by $\left(\frac{i\alpha}{2}\right)^{2N+1} \frac{(4N+1)!}{(2N+1)!}$ and replace $m \to N-n$. We get

$$\sum_{n=0}^{N} \left(\frac{i\alpha}{2}\right)^{2n} \frac{(4N-2n+1)!(2N+1)!}{(4N+1)!(2n)!(2N-2n+1)!} = 0 \tag{40}$$

Since $\lim_{N\to\infty} \frac{(4N-2n+1)!(2N+1)!}{(4N+1)!(2N-2n+1)!} = 2^{-2n}$, $N \to \infty$ limit of the equation (34) is

$$\sum_{n=0} \frac{1}{2n!} (i\alpha)^{2n} = 0. \tag{41}$$

Namely, $\cos\left(\frac{1}{\alpha}\right) = 0$, it means $\alpha = \frac{2}{(2n+1)\pi}$ where n =integer.

Hence, the energy quanta of the present theory are $\frac{m^2 L}{(2n+1)\pi}$ (we can know the power of L by a dimensional analysis) in contrast to one in periodic boundary condition($\frac{m^2 L}{2n\pi}$). It is identical to energy quanta in the anti-periodic case.

6 Conclusion

In this paper, we quantized two-dimensional massive scalar fields in light-cone frame. In the process of quantization, we did not impose periodic boundary condition. Our V-shaped boundary condition is fully consistent to causality. The Fock space can be constructed in this formalism. As a result, we obtained a different spectrum from the case of periodic boundary condition.

The formalism in this paper is applicable to other theories. In particular, an application to gauge theory is interesting. As pointed out in many references (see [4] etc.), the light-cone gauge $A^+ = 0$ is not appropriate under periodic boundary conditions; the zero mode of A^+ cannot be gauged away by gauge transformation ($A^+ \to A^+ + \partial^+\Lambda$). Some vacuum structures, for example, θ-vacuum in the Schwinger model, come from dynamics of this residual zero-mode of A^+. However, the light-cone gauge $A^+ = 0$ is fully consistent to our boundary condition. The vacuum structure may come from other sources. This problem is worth attacking. The most important question is whether our vacuum is stable even in interacting theory, and this will remain a problem in the future.

In this paper, we found a new formalism of light-cone quantization. The author expects that this new formalism will lead to new features of light-cone quantization.

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